

On Glimm's Theorem for locally quasi-compact almost Hausdorff G -spaces

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Abstract. The primitive ideal space $\text{Prim } \mathcal{A}$ of a postliminal C^* -algebra \mathcal{A} is locally quasi-compact and almost Hausdorff. If G is a locally compact group acting strongly continuously on \mathcal{A} by isomorphisms, then $\text{Prim } \mathcal{A}$ becomes a G -space in a natural way. Glimm's intention was to characterize nice behaviour of such G -spaces, namely to find sufficient conditions for $\text{Prim } \mathcal{A} / G$ to be almost Hausdorff. In this note we give a reformulation of Glimm's theorem taking into account some addenda of M. Rieffel. We emphasize topological aspects of the theory and avoid measure theory.

As an application of Glimm's theorem we state: Let \mathcal{A} be separable and postliminal, and suppose that all orbits of the G -space $\text{Prim } \mathcal{A}$ are locally closed. Then for every prime ideal P of the covariance algebra $C^*(G, \mathcal{A})$ there exists a unique orbit $G \cdot Q$ of $\text{Prim } \mathcal{A}$ such that the restriction $\text{res}_{\mathcal{A}}(P)$ of P to \mathcal{A} is equal to the kernel $k(G \cdot Q) = \bigcap_{g \in G} g \cdot Q$ of this orbit.

A modern exposition with complete proofs of this central part of harmonic analysis seems to be justified although these results are not new.

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Introduction

In this note we shall discuss group actions on topological spaces which do not satisfy the Hausdorff separability axiom. Throughout this text all topological groups G and topological spaces X are always understood to be T_0 : If $x \neq y$ are in X such that $x \in \{y\}^-$, then there exists an open x -neighborhood U of X such that $y \notin U$.

We begin with the basic definitions and some elementary results.

Lemma 1. *If G is a topological group (which is T_0), then G is Hausdorff.*

Proof. Let $g_1 \neq g_2$ be in G . Since G is T_0 , there exists a symmetric open e -neighborhood U of G such that $g_1 \notin U g_2$. Choose a symmetric open e -neighborhood W of G such that $W^2 \subset U$. Now it is easy to see that $W g_1$ and $W g_2$ are disjoint open neighborhoods of g_1 and g_2 respectively. \square

A G -space X of a topological group G consists of a (T_0) -space X and a continuous map $G \times X \rightarrow X$ satisfying $g \cdot (h \cdot x) = (gh) \cdot x$ and $e \cdot x = x$ for all $g, h \in G$ and $x \in X$. A G -space is called transitive if $X = G \cdot x$ for one and hence for all $x \in X$. It is well-known that stabilizers are closed, see e.g. Lemma 1 of [1].

Lemma 2 (Robert J. Blattner, 1965).

Let X be a G -space and $x \in X$ arbitrary. Then $G_x = \{g \in G : g \cdot x = x\}$ is a closed subgroup of G .

Proof. Obviously G_x is a subgroup. Let g_λ be a net in G_x which converges to $g \in G$. Then $x = g_\lambda \cdot x \rightarrow g \cdot x$ so that $g \cdot x \in \{x\}^-$. On the other hand, $g \cdot x = g \cdot g_\lambda^{-1} \cdot x \rightarrow x$ so that $x \in \{g \cdot x\}^-$. Since X is T_0 , it follows $g \cdot x = x$. Thus G_x is closed. \square

Lemma 3. *If H is a closed subgroup of a topological group G , then G/H is Hausdorff and the quotient map $\pi : G \rightarrow G/H$ is open.*

Proof. Obviously π is open. Let $g_1, g_2 \in G$ such that $g_1 H \neq g_2 H$. As $g_2 H$ is closed and $g_1 \notin g_2 H$, there is an open e -neighborhood U of G such that $U g_1 \cap g_2 H = \emptyset$. Choose a symmetric open e -neighborhood W of G such that $W^2 \subset U$. Now it is easy to see that $W g_1 H \cap W g_2 H = \emptyset$ so that $\pi(W g_1)$ and $\pi(W g_2)$ are disjoint open neighborhoods of $g_1 H$ and $g_2 H$ in G/H respectively. \square

A topological space X is quasi-compact if every open cover of X admits a finite subcover. We say that X is locally quasi-compact if every x -neighborhood of X contains a quasi-compact x -neighborhood. Further X is (locally) compact if X is (locally) quasi-compact and Hausdorff. If H is a closed subgroup of a (locally) compact group G , then G/H is (locally) compact, too.

It is necessary to distinguish carefully between transitive G -spaces and homogeneous spaces.

Definition 4. Let G be a topological group and X a transitive G -space. We say that X is a homogeneous space if $\omega_x : G/G_x \rightarrow X$, $\omega_x(gG_x) = g \cdot x$ is a homeomorphism for one and hence for all $x \in X$.

Lemma 3 implies that every homogeneous space is Hausdorff.

Let X be a G -space and $x \in X$. The orbit $G \cdot x$ is a transitive G -space. By definition $G \cdot x$ is a homogeneous space if and only if $\omega_x : G/G_x \rightarrow G \cdot x$ is a homeomorphism with respect to the relative topology of X on $G \cdot x$.

Lemma 5. *Let X be a transitive G -space of a topological group G . Then X is a homogeneous space if and only if $N \cdot x$ is an x -neighborhood of X for every e -neighborhood N of G and every $x \in X$.*

In the sequel we will investigate whether the orbit space X/G of a G -space X has nice topological properties. In particular we shall be concerned with the question if orbits are locally closed.

Definition 6. A subset A of a topological space X is said to be locally closed if A is the intersection of a closed and an open subset of X .

This is the case if and only if A is relatively open in its closure, or equivalently, if every $x \in A$ has an open neighborhood U such that $U \cap \overline{A} \subset A$.

Important examples of T_0 -spaces which are not Hausdorff in general are primitive ideal spaces $\text{Prim } \mathcal{A}$ of C^* -algebras \mathcal{A} . (The spectrum $\widehat{\mathcal{A}}$ of \mathcal{A} is in general not T_0 .) It is well-known that $\widehat{\mathcal{A}}$ and hence $\text{Prim } \mathcal{A}$ are locally quasi-compact. The locally closed subsets of $\text{Prim } \mathcal{A}$ are easily characterized: If $A \subset \text{Prim } \mathcal{A}$ is locally closed, then by definition of the Jacobson topology there exist closed ideals I and J of \mathcal{A} such that $A = h(J) \cap (\text{Prim } \mathcal{A} \setminus h(I))$. Thus A is homeomorphic to the primitive ideal space of the subquotient $(I + J)/J$ of \mathcal{A} .

The Open Mapping Theorem

Our aim is to prove that any G -equivariant map of a σ -compact homogeneous space X onto a Baire G -space Y is open. Further we will see that any locally quasi-compact, almost Hausdorff space is Baire.

Definition 7. A topological space X is called almost Hausdorff if every non-empty, closed subset A of X contains a non-empty, relatively open Hausdorff subset.

A singleton in X is simply a one-point subset $\{x\}$ of X . First properties of almost Hausdorff spaces are

Lemma 8. *Let X be an almost Hausdorff space.*

1. *Singletons are locally closed in X . In particular X is T_0 .*

2. **Every** non-empty subset B of X contains a non-empty, relatively open Hausdorff subset, and is hence itself almost Hausdorff.
3. There exists a dense open Hausdorff subset of X .

Proof.

1. Let $x \in X$ be arbitrary. Since X is almost Hausdorff, there exists an open subset U of X such that $U \cap \{x\}^-$ is non-empty and Hausdorff. Obviously $x \in U$ because U is open, and $U \cap \{x\}^- = \{x\}$ because $U \cap \{x\}^-$ is Hausdorff. Thus $\{x\}$ is locally closed.

Let $x, y \in X$ be arbitrary. If $y \notin \{x\}^-$, then we are done because $X \setminus \{x\}^-$ is an open neighborhood of y not containing x . So we can assume $y \in \{x\}^-$. Since $\{x\}$ is locally closed, there exists an open x -neighborhood of X such that $U \cap \{x\}^- = \{x\}$ does not contain y . Hence X is T_0 .

2. Let B be an arbitrary subset of X and A its closure in X . Since X is almost Hausdorff, there is a non-empty, relatively open Hausdorff subset U of A . Then $U \cap B$ is a non-empty, relatively open Hausdorff subset of B .
3. Zorn's Lemma shows that there exists a maximal non-empty, open Hausdorff subset U of X . Suppose that U is not dense in X . Then there exists a non-empty, open Hausdorff subset V of $X \setminus \overline{U}$. Now $U \cup V$ is open in X and Hausdorff, in contradiction to the maximality of U . This proves our claim.

□

We return to our main example: Let \mathcal{A} be a C^* -algebra. We say that \mathcal{A} is liminal if $\pi(\mathcal{A})$ is contained in the C^* -algebra $K(\mathfrak{H})$ of all compact operators for every irreducible $*$ -representation π of \mathcal{A} in a Hilbert space \mathfrak{H} . Further we say that \mathcal{A} is postliminal if any non-zero quotient of \mathcal{A} contains a non-zero liminal ideal. It is well-known that $\widehat{\mathcal{A}} \cong \text{Prim } \mathcal{A}$ is almost Hausdorff provided that \mathcal{A} is postliminal, see e.g. [2]. If \mathcal{A} is liminal, then $\widehat{\mathcal{A}} \cong \text{Prim } \mathcal{A}$ is T_1 . If \mathcal{A} admits a continuous trace, then \mathcal{A} is liminal and $\widehat{\mathcal{A}} \cong \text{Prim } \mathcal{A}$ is Hausdorff.

Next we discuss Baire spaces.

Definition 9. A topological space X is called a Baire space if the intersection $\bigcap_{n=1}^{\infty} U_n$ of any sequence of dense open subsets U_n of X is dense in X .

It is easy to see that X is a Baire space if and only if the union $\bigcup_{n=1}^{\infty} A_n$ of any sequence A_n of closed subsets with empty interior has empty interior. This contraposition of the assertion in Definition 9 is used frequently.

As before let \mathcal{A} be a (not necessarily postliminal) C^* -algebra and $\text{Prim } \mathcal{A}$ its primitive ideal space. Let $P(\mathcal{A})$ denote set of all pure states of \mathcal{A} with the relative topology

of \mathcal{A}' . Choquet has shown that $P(\mathcal{A})$ is a Baire space, see Appendix (B14) of [2]. From this we deduce that any locally closed subset of $\text{Prim } \mathcal{A}$ is a Baire space.

A G_δ -subset of a topological space is a countable intersection of open subsets.

Lemma 10. *If W is a G_δ -subset of a locally compact space X , then W is a Baire space in the relative topology of X .*

Proof. Let U_n be a sequence of open subsets of X such that $U_n \cap W$ is dense in W . Let V be an open subset of X such that $V \cap W \neq \emptyset$. Since W is a G_δ , there exists a sequence W_n of open subsets of X such that $W = \bigcap_{n=1}^\infty W_n$. By induction we will define a decreasing sequence of compact subsets K_n of X such that $\text{int}(K_n) \cap W \neq \emptyset$ and $K_n \subset U_n \cap W_n$: Let $K_0 \subset V$ be compact such that $\text{int}(K_0) \cap W \neq \emptyset$. Suppose that K_0, \dots, K_{n-1} have been chosen. Since $K_{n-1} \cap W \neq \emptyset$ and $U_n \cap W$ is dense in W , it follows that $K_{n-1} \cap U_n \cap W \neq \emptyset$. As X is locally compact, there exists a compact subset $K_n \subset K_{n-1} \cap U_n \cap W_n$ such that $\text{int}(K_n) \cap W \neq \emptyset$. By the finite intersection property we obtain $\bigcap_{n=1}^\infty K_n \neq \emptyset$ and hence $(\bigcap_{n=1}^\infty U_n) \cap V \cap W \neq \emptyset$. For V was arbitrary, we see that $(\bigcap_{n=1}^\infty U_n) \cap W$ is dense in W . \square

Lemma 11. *Let X be a topological space.*

1. *If X is Baire and U is an open subset of X , then U is Baire.*
2. *If U is a dense open subset of X and if U is Baire, then X is Baire.*
3. *If W is a G_δ -subset of a locally quasi-compact, almost Hausdorff space X , then W is Baire.*

Proof.

1. Let U_n be dense and open subsets of U . Then $V_n = U_n \cup (X \setminus \overline{U})$ is dense and open in X . Since X is Baire, it follows that $\bigcap_{n=1}^\infty V_n$ is dense in X . Now it clear that $\bigcap_{n=1}^\infty U_n$ is dense in U .
2. Let U_n be dense and open subsets of X . Then $U_n \cap U$ is dense and open in U . Since U is Baire, the subset $(\bigcap_{n=1}^\infty U_n) \cap U$ is dense in U . It follows that $\bigcap_{n=1}^\infty U_n$ is dense in X .
3. Since X is almost Hausdorff, there exists a dense open Hausdorff subset U of X by part 3. of Lemma 8. Note that U is locally compact. Since $U \cap W$ is a G_δ -subset of U , it follows that $U \cap W$ is Baire by Lemma 10. For $U \cap W$ is dense in W , we see that W is Baire by part 2. of this lemma.

\square

A Hausdorff topological space is σ -compact if it is a countable union of compact subsets.

A fundamental result involving Baire spaces is

Theorem 12 (Open mapping theorem).

Let G be a topological group and $\varphi : X \longrightarrow Y$ a G -equivariant map of a non-empty, σ -compact homogeneous G -space X onto a transitive G -space and Baire space Y . Then φ is an open map. In particular Y is a homogeneous space, too.

Proof. Let $x \in X$ be arbitrary and U an x -neighborhood. We must prove that $\varphi(U)$ is a $\varphi(x)$ -neighborhood. First, by continuity there is an e -neighborhood N of G such that $N \cdot x \subset U$. We can choose a (smaller) open e -neighborhood L of G such that $L^{-1} = L$ and $L^2 \subset N$. Then we know that $L \cdot x$ is an open subset of X because X is a homogeneous space. Since X is σ -compact, there exists a sequence $g_n \in G$ such that $X = \bigcup_{n=0}^{\infty} g_n L \cdot x$. From this we get $Y = \bigcup_{n=0}^{\infty} g_n L \cdot \varphi(y)$ because φ is surjective and G -equivariant. Since Y is a Baire space, some $g_n L \cdot \varphi(y)$ has a non-empty interior. Thus $L \cdot \varphi(y)$ itself has a non-empty interior. Hence there exists an element $g \in L$ such that $g^{-1} L \cdot \varphi(y)$ is a $\varphi(y)$ -neighborhood. Since $g^{-1} L \cdot \varphi(y) \subset N \cdot \varphi(y) \subset \varphi(U)$, we see that $\varphi(U)$ is a $\varphi(x)$ -neighborhood. This proves φ to be open.

Note that $G_x \subset G_{\varphi(x)}$ so that there is a natural open map $\nu_x : G/G_x \longrightarrow G/G_{\varphi(x)}$. The commutative diagram

$$\begin{array}{ccc} G/G_x & \xrightarrow{\nu_x} & G/G_{\varphi(x)} \\ \omega_x \downarrow & & \downarrow \omega_{\varphi(x)} \\ X & \xrightarrow{\varphi} & Y \end{array}$$

shows us that Y is a homogeneous space: Since ν_x and φ are open maps and ω_x is a homeomorphism, it follows that $\omega_{\varphi(x)}$ is a homeomorphism, too. \square

Provided that G itself is σ -compact we deduce

Corollary 13. *If X is a transitive G -space of a σ -compact topological group G and if X is a Baire space, then X is a homogeneous G -space.*

We emphasize that in Theorem 12 we do not assume Y to be Hausdorff. However, the assumption that Y is a transitive G -space and a Baire space implies that Y is a homogeneous space and hence Hausdorff.

Glimm's Theorem

In this section we will give a complete proof of Glimm's theorem which contains necessary and sufficient condition for X/G to be almost Hausdorff. The definition of almost Hausdorff spaces is due to J. Glimm, see [3]. A more distinct exposition can be found in Section 7 of [4] where M. Rieffel proves the following neat result.

Proposition 14 (Marc A. Rieffel, 1979).

Let G be a topological group and X an almost Hausdorff G -space. If $x \in X$ such that $G \cdot x$ is locally compact in the relative topology of X , then $G \cdot x$ is locally closed in X .

Proof. We can assume that $G \cdot x$ is dense in X . Let $y \in G \cdot x$ be arbitrary. Since X is almost Hausdorff, there exists a non-empty open Hausdorff subset U of X . Clearly $U \cap G \cdot x \neq \emptyset$. By translation of U we can achieve $y \in U$. Since $G \cdot x$ is locally compact, there is a compact y -neighborhood V of $G \cdot x$ such that $V \subset U \cap G \cdot x$. Let W be an open subset of U such that $W \cap G \cdot x$ is equal to the interior of V . We shall show that $W \cap \overline{G \cdot x} \subset G \cdot x$: Let z_λ be a net in $G \cdot x$ which converges to $z \in W$. Clearly $z_\lambda \in W \cap G \cdot x \subset V$ for $\lambda \geq \lambda_0$. Since V is compact, z_λ has a subnet converging to $z_0 \in V$. For limits are unique in the Hausdorff subset U , it follows $z = z_0 \in G \cdot x$. This proves $G \cdot x$ to be locally closed. \square

The open mapping theorem and the preceding proposition allow us to characterize the well-behaved orbits of a locally quasi-compact, almost Hausdorff G -space.

Proposition 15 (J. Glimm and M. Rieffel).

Let G be a σ -compact, locally compact group and X a locally quasi-compact, almost Hausdorff G -space. For every $x \in X$ there are equivalent:

1. *The orbit $G \cdot x$ is locally closed in X .*
2. *$G \cdot x$ is a G_δ -subset of its closure.*
3. *$G \cdot x$ is a Baire space in the relative topology of X .*
4. *$G \cdot x$ is a homogeneous space.*
5. *$G \cdot x$ is locally compact in the relative topology of X .*

Proof. If the orbit $G \cdot x$ is locally closed, then $G \cdot x$ is a relatively open subset and hence a G_δ -subset of its closure $\overline{G \cdot x}$. This proves $1. \Rightarrow 2.$ If $G \cdot x$ is a G_δ -subset of $\overline{G \cdot x}$, then $G \cdot x$ is a Baire space by part 3. of Lemma 11 because the closed subspace $\overline{G \cdot x}$ is locally quasi-compact and almost Hausdorff. Thus $2. \Rightarrow 3.$ For $3. \Rightarrow 4.$ we resort to the open mapping theorem using that G is σ -compact. Next we prove $4. \Rightarrow 5.$ Suppose that $G \cdot x$ is a homogeneous space. Since G is locally compact and G_x is a closed subgroup of G by Lemma 2, it follows that $G \cdot x \cong G/G_x$ is locally compact. Finally $5. \Rightarrow 1.$ follows from Proposition 14. \square

In Proposition 15 we encounter two different types of properties: The conditions $G \cdot x$ is locally closed / a G_δ -subset of its closure concern the way in which $G \cdot x$ is situated in the space X and hence the topological relation between different orbits. On the other hand the conditions $G \cdot x$ is a Baire space / a homogeneous space / locally compact are properties of a single orbit.

As a variant of the preceding considerations we state

Lemma 16. *Let G be a σ -compact group and X a G -space such that all closed subspaces of X are Baire. If $x \in X$ such that $G \cdot x$ is locally closed, then $G \cdot x$ is a homogeneous space.*

Proof. Suppose that $G \cdot x$ is locally closed so that this orbit is an open subset of its closure. Since $\overline{G \cdot x}$ is Baire, it follows from part 1. of Lemma 11 that $G \cdot x$ is Baire. For G is σ -compact, Theorem 12 implies that $G \cdot x$ is a homogeneous space. \square

Note that the assumption that closed subspaces of X are Baire does not suffice to prove the implication 2. \Rightarrow 3. of Proposition 15.

The next proposition contains the essential step in the proof of Glimm's theorem. For the convenience of the reader we reproduce Glimm's beautiful argument which can be found in [3].

Proposition 17 (Existence of a slice-like neighborhood).

Let X be a G -space of a locally compact group such that all G -orbits in X are homogeneous spaces. Assume that X is second-countable, locally quasi-compact, and almost Hausdorff. Then

1. *For every non-empty, open subset V of X and every e -neighborhood N of G there exists a non-empty, open subset $U \subset V$ with the following property: If $g \in G$ and if $U_0 \subset U$ is non-empty, open such that $g \cdot U_0 \subset U$, then $N \cdot U_0 \cap g \cdot U_0 \neq \emptyset$.*
2. *For every non-empty, open subset V of X and every e -neighborhood N of G , there exists a non-empty, open subset $U \subset V$ such that $N \cdot x \cap U = G \cdot x \cap U$ for all $x \in U$.*

Proof.

1. Suppose that the assertion of 1. does not hold true so that there is a non-empty, open subset V of X and an e -neighborhood N of G with the following property: For every non-empty, open subset $U \subset V$ there exist an element $g \in G$ and a non-empty, open subset $U_0 \subset U$ such that $g \cdot U_0 \subset U$ and $N \cdot U_0 \cap g \cdot U_0 = \emptyset$. Since X is almost Hausdorff, we can assume that V is Hausdorff. For X is second countable, there is a basis $\{W_n : n \geq 1\}$ of the topology of X .

By induction we can choose elements $g_n \in G$ and compact subsets $E_n \subset V$ with non-empty interior such that the following conditions are satisfied:

- (a) $g_n \cdot E_n \subset E_{n-1}$ and $E_n \subset E_{n-1}$,
- (b) $N \cdot E_n \cap g \cdot E_n = \emptyset$,
- (c) $E_n \subset W_n$ or $E_n \cap W_n = \emptyset$.

We shall explain the details: Assume that E_1, \dots, E_{n-1} are given such that the conditions (a)-(c) hold true. If $E_{n-1} \cap W_n \neq \emptyset$, then W_n has a non-empty intersection with the interior $\overset{\circ}{E}_{n-1}$ of E_{n-1} so that there exist an element $g_n \in G$ and a non-empty, open subset $U_n \subset \overset{\circ}{E}_{n-1} \cap W_n$ such that $g_n \cdot U_n \subset \overset{\circ}{E}_{n-1} \cap W_n$ and $N \cdot U_n \cap g_n \cdot U_n = \emptyset$ by the defining property of V and N . Since X is locally

quasi-compact, we can choose a compact subset $E_n \subset U_n$ with non-empty interior which, by definition, satisfies (a)-(c). On the other hand, if $E_{n-1} \cap W_n = \emptyset$, we can find $g_n \in G$ and $U_n \subset \overset{\circ}{E}_{n-1}$ non-empty, open such that $N \cdot U_n \cap g \cdot U_n = \emptyset$. Now any compact subset $E_n \subset U_n$ with non-empty interior satisfies our requirements.

Since E_1 is compact and E_n is a decreasing sequence of non-empty closed subsets of E_1 , there exists a point $x \in \bigcap_{n=1}^{\infty} E_n$ by the finite intersection property. Now condition (a) and (c) imply $g_n \cdot x \rightarrow x$, and (b) implies $g_n \cdot x \notin N \cdot x$. Thus $N \cdot x$ cannot be an x -neighborhood of $G \cdot x$, in contradiction to the assumption that all G -orbits are homogeneous spaces. This proves assertion 1.

2. Given N and V , we fix a non-empty, open Hausdorff subset $W_2 \subset V$, which is possible because X is almost Hausdorff. Further we choose a non-empty, open subset $W_1 \subset W_2$ and a compact e -neighborhood $L \subset N$ of G such that $L \cdot W_1 \subset W_2$. Here we use the fact that G is locally compact. Further it follows from part 1. that there exist a non-empty, open subset $U \subset W_1$ with the following property: If $g \in G$ and $U_0 \subset U$ is non-empty, open such that $g \cdot U_0 \subset U$, then it follows $L \cdot U_0 \cap g \cdot U_0 \neq \emptyset$.

Let $x \in U$ be arbitrary. We must prove $G \cdot x \cap U \subset L \cdot x \cap U$. Let $g \in G$ be arbitrary such that $g \cdot x \in U$. If $\{U_n : n \geq 1\}$ is a basis of x -neighborhoods, then $g \cdot U_n \subset U$ for large n , and thus $L \cdot U_n \cap g \cdot U_n \neq \emptyset$ by definition of U . Hence there exist $h_n \in L$ and $x_n, y_n \in U_n$ such that $g \cdot x_n = h_n \cdot y_n$. Since L is compact, we can assume that $h_n \rightarrow h$ converges in L . Since W_2 is Hausdorff and $x_n, y_n \rightarrow x$, we see that

$$g \cdot x = \lim_{n \rightarrow \infty} g \cdot x_n = \lim_{n \rightarrow \infty} h_n \cdot y_n = h \cdot x$$

lies in $L \cdot x \cap U$, which completes the proof.

□

We shall give a further explanation of condition 2: For any open subset V of X and any e -neighborhood N of G there is an open subset U of V such that $N \cdot x \cap U = G \cdot x \cap U$ for every $x \in U$. In Glimm's own words: N acts locally as transitive as G does. One should think of U as a prolate open subset which is transversal to the orbits passing through V and whose width becomes arbitrarily small as N shrinks to $\{e\}$. The orbit picture in U is similar to that close to a slice through a principal point (of a proper G -space).

Now we present a reformulation of Glimm's theorem characterizing nice behaviour of G -spaces, see Theorem 1 of [3]. In contrast to the original proof we shall avoid measure theoretical arguments involving the Borel structure of the orbit space X/G and quasi-invariant ergodic measures on X .

Theorem 18 (James Glimm, 1961).

Let G be a σ -compact, locally compact group and X a second countable, locally quasi-compact, almost Hausdorff G -space. Then there are equivalent:

1. X/G is almost Hausdorff.
2. X/G is a T_0 -space.
3. Every orbit is a homogeneous space.

Proof. The implication 1. \Rightarrow 2. follows from part 1. of Lemma 8. For 2. \Rightarrow 3. we assume that X/G is a T_0 -space. Let $\pi : X \twoheadrightarrow X/G$ denote the projection onto the orbit space which is an open map. Since X and hence X/G are first countable, the point $G \cdot x$ of X/G has a countable neighborhood basis \dot{U}_n . For X/G is T_0 , we obtain $\{G \cdot x\} = \bigcap_{n=1}^{\infty} \dot{U}_n \cap \{G \cdot x\}^-$. This implies that

$$G \cdot x = \bigcap_{n=1}^{\infty} \pi^{-1}(\dot{U}_n) \cap \overline{G \cdot x}$$

is a G_δ -subset of its closure. As G is σ -compact and X is locally quasi-compact and almost Hausdorff, Proposition 15 implies that $G \cdot x$ is a homogeneous space.

Next we verify 3. \Rightarrow 1. We must prove that any non-empty closed subset \dot{A} of the orbit space X/G contains a non-empty relatively open Hausdorff subset \dot{W} . Since $A = \pi^{-1}(\dot{A})$ is G -invariant and closed, it is clear that A is a second countable, locally quasi-compact, almost Hausdorff G -space satisfying 3. First we choose a non-empty relatively open Hausdorff subset W_2 of A . Fix a compact e -neighborhood N of G and a non-empty relatively open subset W_1 of W_2 such that $N \cdot W_1 \subset W_2$. By applying Proposition 17 to the G -space A we find a non-empty relatively open subset U of W_1 such that $N \cdot x \cap U = G \cdot x \cap U$ for all $x \in U$. Finally we define $W = G \cdot U$ which is relatively open in A . Suppose that $\dot{W} = \pi(W)$ is not Hausdorff so that there exist points $x, y \in W$ such that $G \cdot x \neq G \cdot y$ cannot be separated by disjoint G -invariant open neighborhoods. For X is first countable, we find open sets $X_n, Y_n \subset U$ such that $\{x\} = \bigcap_{n=1}^{\infty} X_n$ and $\{y\} = \bigcap_{n=1}^{\infty} Y_n$. By assumption we have $G \cdot X_n \cap G \cdot Y_n \neq \emptyset$ so that there exist elements $x_n \in X_n$, $y_n \in Y_n$, $g_n \in G$ such that $x_n = g_n \cdot y_n$. Using $N \cdot y_n \cap U = G \cdot y_n \cap U$ we can even find $h_n \in N$ such that $x_n = h_n \cdot y_n$. Since N is compact, we can assume that $h_n \rightarrow h$ converges in N . This implies

$$x = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} h_n \cdot y_n = h \cdot y$$

and hence $G \cdot x = G \cdot y$, a contradiction. Thus $\dot{W} = \pi(W)$ must be Hausdorff. This proves X/G to be almost Hausdorff. The proof of Glimm's theorem is complete. \square

In retrospect, Glimm's merit is the ingenious proof of Proposition 17 which implies that X/G is almost Hausdorff provided that all orbits are homogeneous spaces.

Lemma 19. *Let G be a σ -compact group and X a G -space such that all closed subspaces of X are Baire. If X/G is almost Hausdorff, then every orbit is a homogeneous space.*

Proof. Suppose that X/G is almost Hausdorff. By part 1. of Lemma 8 singletons are locally closed in X/G so that $\{G \cdot x\}$ is relatively open in $\{G \cdot x\}^-$. Let $\pi : X \twoheadrightarrow X/G$ be the projection. Then $G \cdot x = \pi^{-1}(\{G \cdot x\})$ is relatively open in its closure $\overline{G \cdot x} \subset \pi^{-1}(\{G \cdot x\}^-)$ which means that $G \cdot x$ is locally closed. Now Lemma 16 implies that $G \cdot x$ is a homogeneous space. \square

Glimm's motivation was to study group actions on primitive ideal spaces of C^* -algebras, which are non-Hausdorff in general: Let (G, \mathcal{A}) be a covariance system, i.e., a locally compact group G acting strongly continuous as a group of isomorphisms on a (not necessarily postliminal) C^* -algebra \mathcal{A} . If $X = \text{Prim } \mathcal{A}$ is endowed with the Jacobson topology, then $\text{Prim } \mathcal{A}$ is locally quasi-compact and closed subspaces of $\text{Prim } \mathcal{A}$ are Baire by Choquet's theorem. Further $\text{Prim } \mathcal{A}$ becomes a G -space in a natural way. Consequently Lemma 19 applies: If G is σ -compact and $\text{Prim } \mathcal{A} / G$ is almost Hausdorff, then all orbits in $\text{Prim } \mathcal{A}$ are homogeneous spaces. If in addition \mathcal{A} is separable and postliminal, then $\text{Prim } \mathcal{A}$ is second countable and almost Hausdorff. In this case the implication 3. \Rightarrow 1. of Theorem 18 holds true: If all orbits in $\text{Prim } \mathcal{A}$ are locally closed, then $\text{Prim } \mathcal{A} / G$ is almost Hausdorff.

Applications in harmonic analysis

Next we prove a variant of the following well-known result: Let N be a second countable closed normal subgroup of a locally compact group G . Since G acts on N and hence on $C^*(N)$ by conjugation, \widehat{N} becomes a G -space. Suppose that N is of type I and that all orbits of \widehat{N} are locally closed. Then for every irreducible representation π of G there exists a unique orbit $G \cdot \sigma$ of \widehat{N} such that $\pi|_N$ is weakly equivalent to $G \cdot \sigma$.

Let (G, \mathcal{A}) be a covariance system. The associated C^* -covariance algebra $C^*(G, \mathcal{A})$ is defined as follows: Let $\mathcal{C}_0(G, \mathcal{A})$ be the $*$ -algebra of \mathcal{A} -valued continuous functions with compact support, with norm

$$\|f\|_1 = \int_G |f(x)| \, dx,$$

involution

$$f^*(x) = \Delta_G(x^{-1}) (f(x^{-1})^*)^x,$$

and multiplication

$$(f * g)(x) = \int_G f(xy)^{y^{-1}} g(y^{-1}) \, dy.$$

Then $C^*(G, \mathcal{A})$ is defined as the completion of $\mathcal{C}_0(G, \mathcal{A})$ with respect to the norm

$$\|f\|_* = \sup \{ \|\pi(f)\| : \pi \text{ is a bounded } * \text{-representation of } \mathcal{C}_0(G, \mathcal{A}) \}.$$

Note that \mathcal{A} acts on $C^*(G, \mathcal{A})$ by $(a * f)(x) = a^x f(x)$ and $(f * a)(x) = f(x)a$ as an algebra of left and right multipliers respectively. In particular $a * (b * f) = (a * f) * g$, $(ab) * f = a * (b * f)$, and $(a * f)^* = f^* * a^*$. Furthermore G acts isometrically and strongly continuously on $C^*(G, \mathcal{A})$ by left multiplication $\lambda(z)f(x) = f(z^{-1}x)$, right multiplication $\rho(z)f(x) = \Delta_G(z)f(xz)^{z^{-1}}$, and conjugation $f^z(x) = \Delta_G(z^{-1})f(xz^{-1})^z$. The operators $\lambda(z)$ and $\rho(z)$ are left and right multipliers respectively. Note that $(f * g)^z = f^z * g^z$ and $(a * f)^z = a^z * f^z$ for all $f, g \in C^*(G, \mathcal{A})$, $a \in \mathcal{A}$, and $z \in G$.

Let I be a two-sided closed ideal of $C^*(G, \mathcal{A})$. Using that $C^*(G, \mathcal{A})$ has approximate identities we conclude $\mathcal{A} * I \subset I$ and $I * \mathcal{A} \subset I$. Further I turns out to be $\lambda(G)$ - and $\rho(G)$ -invariant which implies $I^z = I$ for all $z \in G$.

In the sequel all ideals are assumed to be two-sided and closed.

For every ideal I of $C^*(G, \mathcal{A})$ we define

$$\text{res}(I) = \{a \in \mathcal{A} : a * C^*(G, \mathcal{A}) \subset I\},$$

the restriction of I to \mathcal{A} . Clearly $\text{res}(I)$ is a closed, two-sided, and G -invariant ideal of \mathcal{A} because $a^z * C^*(G, \mathcal{A}) = (a * C^*(G, \mathcal{A}))^z \subset I^z = I$ for all $a \in \text{res}(I)$ and $z \in G$.

Conversely, if J is an ideal of \mathcal{A} , then the extension of J from \mathcal{A} up to $C^*(G, \mathcal{A})$ is given by

$$\text{ext}(J) = (C^*(G, \mathcal{A}) * J * C^*(G, \mathcal{A}))^-.$$

Note that $\text{ext}(J)$ is the smallest ideal I of $C^*(G, \mathcal{A})$ which satisfies $J \subset \text{res}(I)$. Clearly $J \subset \text{res}(\text{ext}(J))$ for all ideals J of \mathcal{A} , and $\text{ext}(\text{res}(I)) \subset I$ for all ideals I of $C^*(G, \mathcal{A})$.

Lemma 20. *Let J be a G -invariant ideal of \mathcal{A} . Then $C^*(G, \mathcal{A}) * J$ is contained in the closure of $J * C^*(G, \mathcal{A})$, and $J * C^*(G, \mathcal{A})$ in the closure of $C^*(G, \mathcal{A}) * J$.*

Proof. Let u_λ be an approximating identity of J , i.e., a net in J such that $u_\lambda^* = u_\lambda$, $|u_\lambda| \leq 1$, and $|b - u_\lambda b| \rightarrow 0$ for every $b \in J$. Let $f \in \mathcal{C}_0(G, \mathcal{A})$ and $a \in J$ be arbitrary. Since J is G -invariant, it follows

$$|f(x)a - u_\lambda^x f(x)a| = |(f(x)a)^{x^{-1}} - u_\lambda (f(x)a)^{x^{-1}}| \rightarrow 0$$

for every $x \in G$. Thus

$$|f * a - u_\lambda * (f * a)|_1 = \int_G |f(x)a - u_\lambda^x f(x)a| dx \rightarrow 0$$

by the dominated convergence theorem because the integrand is continuous, bounded, and of compact support in x . Since $u_\lambda * (f * a)$ is in $J * C^*(G, \mathcal{A})$, this proves the first assertion. The second one can be verified similarly. \square

Further $\text{ext}(J)$ coincides with the closure of $\mathcal{C}_0(G, J)$ in $C^*(G, \mathcal{A})$ for G -invariant J .

An ideal I of $C^*(G, \mathcal{A})$ is said to be prime if $I_1 * I_2 \subset I$ implies $I_1 \subset I$ or $I_2 \subset I$ for all ideals I_1 and I_2 of $C^*(G, \mathcal{A})$.

Lemma 21. *If π is a factor representation of a C^* -algebra \mathcal{B} , then $\ker \pi$ is prime.*

Proof. Let I_1 and I_2 be ideals of \mathcal{B} such that $I_1 I_2 \subset \ker \pi$ and $I_2 \not\subset \ker \pi$. Then $\pi(I_2)\mathfrak{H}$ is a non-zero closed subspace of the representation space \mathfrak{H} of π which is $\pi(\mathcal{B})$ - and $\pi(\mathcal{B})'$ -invariant. If P denotes the orthogonal projection onto the subspace $\pi(I_2)\mathfrak{H}$, then P is non-zero and in $\pi(\mathcal{B})' \cap \pi(\mathcal{B})'' = \mathbb{C} \cdot \text{Id}$ which is trivial because π is a factor representation. Thus $P = \text{Id}$ which means $\pi(I_2)\mathfrak{H} = \mathfrak{H}$. Since $\pi(I_1)\pi(I_2) = \pi(I_1 I_2) = 0$, it follows $I_1 \subset \ker \pi$. The proof is complete. \square

A G -invariant ideal J of \mathcal{A} is called G -prime if $J_1 J_2 \subset J$ implies $J_1 \subset J$ or $J_2 \subset J$ for all G -invariant ideals J_1 and J_2 of \mathcal{A} .

The hull $h(I)$ of a G -invariant ideal I of \mathcal{A} is a closed, G -invariant subset of $\text{Prim } \mathcal{A}$, and the kernel $k(A)$ of a G -invariant subset A of $\text{Prim } \mathcal{A}$ is a G -invariant ideal of \mathcal{A} . In order to describe the hull of G -prime ideals we state

Definition 22. A topological space X is called irreducible if $X = A \cup B$ with closed subsets A, B of X implies $X = A$ or $X = B$.

A point x of a topological space is generic if $X = \{x\}^-$. These definitions are borrowed from algebraic geometry.

Lemma 23. *Let X be an irreducible almost Hausdorff space. Then X has a unique generic point.*

Proof. By part 3. of Lemma 8 we find a non-empty dense open Hausdorff subset U of X . Suppose that U contains two distinct point x and y . Since U is Hausdorff, there exist two disjoint open subsets U_1 and U_2 of X such that $x \in U_1$ and $y \in U_2$. Now it follows $X = (X \setminus U_1) \cup (X \setminus U_2)$ in contradiction to the irreducibility of X . Thus $U = \{x\}$ for some $x \in X$. Clearly x is a generic point because U is dense, and x is unique because U is open. \square

In order to prepare the proof of Theorem 25 we note

Proposition 24. *Let (G, \mathcal{A}) be a covariance system.*

1. *If P is a prime ideal of $C^*(G, \mathcal{A})$, then $J = \text{res}(P)$ is a G -prime ideal of \mathcal{A} .*
2. *If J is a G -prime ideal of \mathcal{A} , then $h(J)/G$ is an irreducible subset of $\text{Prim } \mathcal{A}/G$.*
3. *If $\text{Prim } \mathcal{A}/G$ is almost Hausdorff, then for every G -prime ideal J of \mathcal{A} there exists a unique orbit $G \cdot Q$ of $\text{Prim } \mathcal{A}$ such that $J = k(G \cdot Q)$.*

Proof.

1. Let J_1, J_2 be G -invariant ideals of \mathcal{A} such that $J_1 J_2 \subset J$. Now Lemma 20 implies $\text{ext}(J_1) \text{ext}(J_2) = \text{ext}(J_1 J_2) \subset \text{ext}(J) \subset P$. Since P is prime, we conclude that $\text{ext}(J_1) \subset P$ or $\text{ext}(J_2) \subset P$. Consequently $J_1 \subset \text{res}(\text{ext}(J_1)) \subset J$ or $J_2 \subset \text{res}(\text{ext}(J_2)) \subset J$. This proves J to be G -prime.
2. Let A_1, A_2 be closed, G -invariant subsets of $h(J)$ such that $A_1 \cup A_2 = h(J)$. Clearly $J_1 = k(A_1)$ and $J_2 = k(A_2)$ are G -invariant ideals. Further $J_1 J_2 \subset J_1 \cap J_2 = J$. Since J is G -prime, we conclude $J_1 = J$ or $J_2 = J$, and hence $A_1 = h(J)$ or $A_2 = h(J)$. Thus $h(J)/G$ is irreducible.
3. Since J is G -prime and $\text{Prim } \mathcal{A}/G$ is almost Hausdorff, we know that $h(J)/G$ is irreducible and almost Hausdorff by part 2. of this lemma. Now Lemma 23 implies that $h(J)/G$ has a unique generic point $G \cdot Q$ so that $h(J)/G = \{G \cdot Q\}^-$. This yields

$$h(J) = p^{-1}(h(J)/G) = p^{-1}(\{G \cdot Q\}^-) = \overline{G \cdot Q}$$

and hence $J = k(G \cdot Q)$. Here $p : X \twoheadrightarrow X/G$ denotes the projection which is an open map and hence satisfies $p^{-1}(\dot{A})^- = p^{-1}(\dot{A}^-)$ for all subsets \dot{A} of X/G .

□

Theorem 25. *Let (G, \mathcal{A}) be a covariance system where G is a σ -compact, locally compact group and \mathcal{A} a separable postliminal C^* -algebra. Assume that all orbits in $\text{Prim } \mathcal{A}$ are locally closed. Then for every prime ideal P of $C^*(G, \mathcal{A})$ there exists a unique orbit $G \cdot Q$ of $\text{Prim } \mathcal{A}$ such that $\text{res}(P) = k(G \cdot Q)$.*

Proof. First we note that $\text{Prim } \mathcal{A}$ is a second countable, locally quasi-compact, almost Hausdorff G -space because \mathcal{A} is a separable and postliminal C^* -algebra. Since all orbits of $\text{Prim } \mathcal{A}$ are assumed to be locally closed and hence homogeneous spaces by Proposition 15, it follows from Glimm's Theorem that $\text{Prim } \mathcal{A}/G$ is almost Hausdorff. Finally Proposition 24 yields the desired result. □

Actions of compact groups

In this section we are interested in covariance systems (G, \mathcal{A}) with G a compact group. We begin with a technical lemma.

Lemma 26. *Let G be a compact group and X a G -space which contains a dense open Hausdorff subset U . Then there exists a non-empty G -invariant open Hausdorff subset W of X .*

Proof. Let x_0 be in U . Since $G \cdot x_0$ is quasi-compact, there are $g_1, \dots, g_n \in G$ such that $W_0 = \bigcup_{k=1}^n g_k \cdot U$ contains $G \cdot x_0$. Further there exists an open x_0 -neighborhood V_0 such that $G \cdot V_0 \subset W_0$ because W_0 is open and G is compact. As $\bigcap_{k=1}^n g_k \cdot U$ is dense in X ,

it follows that $V = V_0 \cap (\bigcap_{k=1}^n g_k \cdot U)$ is non-empty and open. Clearly $W = G \cdot V \neq \emptyset$ is G -invariant and open. We prove that W is Hausdorff: Let $y_1 \neq y_2$ be in W . Then $y_1 = g \cdot x_1$ for some $x_1 \in V$ and $g \in G$, and $x_2 = g^{-1} \cdot y_2 \in g_k \cdot U$ for some $1 \leq k \leq n$. Since $g_k \cdot U$ is Hausdorff and $V \subset g_k \cdot U$, there are disjoint open neighborhoods X_1 and X_2 of x_1 and x_2 respectively. Finally $g \cdot X_1$ and $g \cdot X_2$ are disjoint open neighborhoods of y_1 and y_2 which proves W to be Hausdorff. \square

Now we are able to prove

Lemma 27. *Let G be a compact group and X an almost Hausdorff G -space. Then all orbits are homogeneous spaces and locally closed in X . Further X/G is almost Hausdorff.*

Proof. First we prove that X/G is almost Hausdorff: Let A be a G -invariant closed subset of X . Clearly A contains a relatively open dense Hausdorff subset U . By the preceding lemma it follows that A contains a non-empty, G -invariant, relatively open Hausdorff subset W . We claim that W/G is Hausdorff: If not, then there are points x and y of W such that $G \cdot x \neq G \cdot y$ cannot be separated by disjoint G -invariant neighborhoods. We choose nets X_λ and Y_λ of neighborhoods of x and y respectively such that $\{x\} = \bigcap_{\lambda \in \Lambda} X_\lambda$ and $\{y\} = \bigcap_{\lambda \in \Lambda} Y_\lambda$. For every $\lambda \in \Lambda$ we find $x_\lambda \in X_\lambda$, $y_\lambda \in Y_\lambda$, and $g_\lambda \in G$ such that $x_\lambda = g_\lambda \cdot y_\lambda$ because $G \cdot X_\lambda \cap G \cdot Y_\lambda \neq \emptyset$. Since G is compact, there is a subnet g_{λ_ν} which converges to $g \in G$. Using that limits are unique in the Hausdorff subset W we conclude

$$x = \lim_{\nu} x_{\lambda_\nu} = \lim_{\nu} g_{\lambda_\nu} \cdot y_{\lambda_\nu} = g \cdot y$$

and hence $G \cdot x = G \cdot y$, a contradiction.

Finally we verify that all orbits are homogeneous spaces. Let $x \in X$ be arbitrary. Since X/G is almost Hausdorff, there exists a non-empty, G -invariant, relatively open Hausdorff subset W of $\overline{G \cdot x}$ such that W/G is Hausdorff. This implies that $G \cdot x = W$ is Hausdorff and locally closed. In particular $G \cdot x$ is a homogeneous space. \square

As a consequence of these results and Proposition 24 we obtain the following result.

Corollary 28. *Let (G, \mathcal{A}) be a covariance system where G is a compact group and \mathcal{A} is a C^* -algebra such that $\text{Prim } \mathcal{A}$ is almost Hausdorff. Then for every prime ideal P of $C^*(G, \mathcal{A})$ there exists a unique orbit $G \cdot Q$ of $\text{Prim } \mathcal{A}$ such that $\text{res}(P) = k(G \cdot Q)$.*

References

- [1] Robert J. Blattner. Group extension representations and the structure space. *Pac. J. Math.*, 15:1101–1113, 1965.
- [2] Jacques Dixmier. *Les C^* -algèbres et leurs représentations*. Deuxième édition. Cahiers Scientifiques, Fasc. XXIX. Gauthier-Villars Éditeur, Paris, 1969.

- [3] James Glimm. Locally compact transformation groups. *Trans. Am. Math. Soc.*, 101:124–138, 1961.
- [4] Marc A. Rieffel. Unitary representations of group extensions; an algebraic approach to the theory of Mackey and Blattner. In *Studies in analysis*, volume 4 of *Adv. in Math. Supl. Stud.*, pages 43–82. Academic Press, New York, 1979.